arXiv:1104.5440v4 [math.SG] 10 May 2012

ON CONFIGURATION SPACES OF STABLE MAPS

YASHA SAVELYEV

[1]

Abstract. We study here some aspects of the topology of the space of smooth, stable, genus 0 curves in a Riemannian manifold X, i.e. the Kontsevich stable curves, which are not necessarily holomorphic. We use the Hofer-Wysocki-Zehnder polyfold structure on this space and some natural characteristic classes, to show that for $X = BU$, the rational homology of the spherical mapping space injects into the rational homology of the space of stable curves. We also give here a definition of what we call q-complete symplectic manifolds, which roughly speaking means Gromov-Witten theory captures all information about homology of the space of smooth stable maps.

1. Introduction

The space of smooth stable curves (unparametrized stable maps) of a Riemann surface into a Riemannian manifold appears naturally in the context Gromov-Witten theory in symplectic geometry, particularly in the context of the beautiful polyfold approach of Hofer-Wysocki-Zehnder [7].

The topology of the configuration space of stable curves in a general Riemannian manifold seems very interesting on its own merit. For example we show that the space of based stable curves has the structure of an H-space. This is interesting as the space of unparametrized based spheres in X does not have an H-space structure.

Moreover, this configuration space may also be very natural in the study of gradient flow for the energy functional on the space of smooth maps of say a Riemann sphere into a Riemannian manifold X. It was first observed by Sacks-Uhlenbeck [12] that the flow lines of the resulting parabolic flow often do not converge to smooth maps but rather the associated maps develop bubbling phenomena. This of course presents problems for Morse theory considerations. For example, Eells and Wood [2] show that in a simply connected Kahler manifold X, the only critical points of the energy functional on the mapping space of a Riemann sphere are (anti)-holomorphic maps, which are also absolutely energy minimizing. If Morse theory worked as expected we could conclude that the topology of the mapping space coincides with the topology of the space of absolute minima, i.e. with the space of (anti)-holomorphic maps, which is usually the wrong conclusion. However in a series of remarkable papers [1], [6], [5], [8], [4] (sorry for incomplete list) it is shown that the conclusion becomes essentially correct after a suitable process of stabilization. One possibility for treating or at least understanding this problem is to partially compactify the space by adding all appropriate stable maps, in such a way that bubbling becomes built in. One may then hope that the gradient flow

[1]2010 Mathematics Subject Classification: 53D45, 57R19

on the enlarged space satisfies some version of Palais-Smale condition, after appropriate completion. It would be most exciting to see if the polyfold theory helps in this.

Our main result concerns injectivity of the map from rational homology of the spherical mapping space into the rational homology of the space of stable curves in the case of $X = BU(r)$. This at least tells us the space of stable curves in that case is topologically non-trivial. Although we are far from understanding the topology of this space.

Finally, we define a notion of q-complete symplectic manifolds, inspired by work Cohen-Segal-Jones [1] studying the spherical mapping space into an almost Kahler manifold by means of certain stabilization of holomorphic spherical mapping spaces.

1.1. **The space of smooth stable maps.**

Let $\mathcal{P}^A_{0,n}\,X$ denote the space whose elements are continuous maps into a Riemannian manifold X, g, with total homology class A, from a nodal connected Riemann surface Σ with genus 0, with complex structure j and n marked points. These maps are assumed smooth on each component of Σ, and satisfy some additional conditions:

- If the restriction u_α of u to a component α is non-constant then u_α is not null-homologous.
- If u_α is constant then the number of special points is at least 3.

We will distinguish one marked point by z_0. A pair of stable maps

$$u_1 : (\Sigma_1, j_1, \{z_i^1\}) \to X, u_2 : (\Sigma_2, j_2, \{z_i^2\}) \to X$$

are *equivalent* if there is a continuous map $\phi : \Sigma_1 \to \Sigma_2$, which is a diffeomorphism on each component, satisfies $\phi^* j_2 = j_1$ maps marked points to marked points and satisfies $u_2(z_i^2) = u_1(z_i^1)$. The quotient by the equivalence relation will be denoted by $\overline{\Omega}^A_n\,X$, and its elements are called stable curves in X.

The topology on $\mathcal{P}^A_{0,n}\,X$ we consider is the standard Gromov topology, (taking into account energy of maps for the metric g) and the arguments of [10, Chapter 5] can be adopted to show that this induces a metrizable topology on $\overline{\Omega}^A_n$. From now on we will suppress mention of the axillary metric g, since topology of our spaces is independent of g. This topology should be equivalent to the topology considered in [7, Section 3.2]. (At any rate the topology in [7] is the one we need for Theorem 1.3.)

Remark 1.1. *Note that under these conditions for a compact Riemannain manifold X a bound on Riemannian energy of u gives a bound on the number of smooth components. As by assumption that each u_α is not null-homologous, it's area and consequently energy is bounded by some constant $\hbar_g$ depending only on g. This is an elementary application of the main compactness theorem for rectifiable currents in geometric measure theory, [3]. This brings up a question: are closed bounded subsets in $\overline{\Omega}^A_n\,X$ compact for a compact X?*

Set

$$\overline{\Omega}^A\,X = colim\,\overline{\Omega}^A_n\,X,$$

the colimit with the maps in the directed system

$$i_n : \overline{\Omega}_n \to \overline{\Omega}_{n+1} \tag{1.1}$$

defined as follows. For $u \in \mathcal{P}^A_{0,n}\, X$, $u : (\Sigma, j, \{z_i\}) \to X$, $i_n(u)$ is an equivalence class of a map u' from

$$\Sigma' = \Sigma \sqcup (\mathbb{CP}^1, 0, z_1', z_2')/z_0 \sim z_1'.$$

The map $i_n(u)$ is u on the component Σ and constant on the new component. The new distinguished marked point z_0' on Σ' is the marked point 0.

It will later be necessary to remember more structure and consider the spaces $\overline{\Omega}_n\, X$ as polyfolds, which is an orbifold version of M-polyfolds, or with even more structure as polyfold groupoids. In this case instead of saying that stable maps are smooth, we should instead require some regularity on the underlying continuous maps, for details see [7, Section 1.1].

For the definition and construction of these structures on spaces of stable maps of the kind we use, the reader is referred to Hofer-Wisocki-Zehnder [7]. Although [7] constructs polyfold structures on spaces of stable maps in a symplectic manifold, the construction clearly works word for word in the above setting. Although it might be at the moment mysterious why it is interesting. The present note is an attempt to explain this.

1.1.1. *Product operation.* Let

$$u_1, u_2 : (\Sigma_1, z_0^1), (\Sigma_2, z_0^2) \to X$$

be representatives for a pair of elements $|u_i|$ in $\overline{\Omega}_n\, X$, respectively $\overline{\Omega}_m\, X$ such that $u_1(z_0^1) = u_2(z_0^2)$. Then we have a product $|u_1| \star |u_2| \in \overline{\Omega}_{n+m-1}\, X$ defined as an equivalence class of a map from

$$\Sigma' = \Sigma_1 \sqcup \Sigma_2 \sqcup (\mathbb{CP}^1, 0, z_1', z_2')/z_0^1 \sim z_1', z_0^2 \sim z_2',$$

which is u_1 respectively u_2 on the components Σ_1, respectively Σ_2 and is constant on the new $\mathbb{CP}^1$ component. The new distinguished marked point for Σ' is 0. In other words we concatenate u_1, u_2 with a ghost bubble as intermediary.

This multiplication is homotopy associative, since the main ghost components of domains for $(u_1 \star u_2) \star u_3$, and $u_1 \star (u_2 \star u_3)$, where concatenation takes place correspond to a pair of points in $\overline{M}_{0,4}$, which we may connect by a path. In other words it is conceptually the same argument as the argument for associativity of quantum multiplications. Also the maps

$$\overline{\Omega}_n\, X \times \overline{\Omega}_m\, X \to \overline{\Omega}_{n+m-1}\, X \xrightarrow{i_{n+m} \circ i_{n+m-1}} \overline{\Omega}_{n+m+1}\, X$$

$$\overline{\Omega}_n\, X \times \overline{\Omega}_m\, X \xrightarrow{i_n \times i_m} \overline{\Omega}_{n+1}\, X \times \overline{\Omega}_{m+1}\, X \to \overline{\Omega}_{n+m+1}\, X,$$

are homotopy equivalent by a similar argument. Consequently there is an induced map in the homotopy category $\overline{\Omega}\, X \times \overline{\Omega}\, X \to \overline{\Omega}\, X$.

If we ask that our maps u are based, i.e. map the distinguished marked point z_0 to $x_0 \in X$, then the corresponding space $\overline{\Omega}_{x_0}\, X$ is a homotopy associative H-space. Consequently homology of $\overline{\Omega}_{x_0}\, X$ is a ring with Pontryagin product.

Notation 1.2. *From now on we will be in the above based situation and the subscript x_0 in $\overline{\Omega}_{x_0}$ will be dropped.*

Our main observation in this paper is this:

Theorem 1.3. *The natural map $H_*(\Omega^2 BSU(r), \mathbb{Q}) \to Cob_*^{orb}(\overline{\Omega}\, BSU(r), \mathbb{Q})$, is injective for $* \leq 2r - 2$.*

On the right hand side we have orbifold bordism groups which are to be defined.

1.2. **Complete symplectic manifolds.** Suppose now (X, ω) is a symplectic manifold. Let $a_i : D_i \to X$, $a_\xi : D_\xi \to \overline{M}_{0,n}$ be smooth maps of closed oriented smooth manifolds, with $\overline{M}_{0,n}$ denoting the moduli space of stable genus 0 Riemann surfaces with n marked points.

Under suitable conditions, for example if (X, ω) is semi-positive we have natural cycles $gw : \overline{\mathcal{M}}_n(A, \{a_i\}, a_\xi) \to \overline{\Omega}\, X$, defined as follows. Consider the diagram below:

$$\begin{tikzcd} & (\prod_i D_i) \times D_\xi \arrow[d, "prod"] \\ \overline{\mathcal{M}}^A_{0,n}(X, J) \arrow[r] & X^n \times \overline{M}_{0,n}, \end{tikzcd} \tag{1.2}$$

with $\overline{\mathcal{M}}^A_{0,n}(X, J)$ denoting the compactified moduli space of genus zero, class A, J-holomorphic curves in X, for a regular ω-tamed J. After perturbing the maps to be transverse, we define $\overline{\mathcal{M}}^A_n(X, \{a_i\}, \xi)$ as the pull-back of this diagram (oriented fibre product) and the cycle gw is defined to be the composition of the projection of $\overline{\mathcal{M}}^A_n(X, \{a_i\}, \xi)$ to $\overline{\mathcal{M}}^A_{0,n}(X; J)$, with the tautological map

$$\overline{\mathcal{M}}^A_{0,n}(X, J) \to \overline{\Omega}\, X.$$

For a completely general symplectic manifold (M, ω) the homology class of the cycle gw is defined via the homology pushforward of the orbifold virtual fundamental class of $\overline{\mathcal{M}}_n(A, \{a_i\}, \xi)$. We shall call cycles gw: ***Gromov-Witten*** cycles. We will also call all 0-dimensional cycles into $\overline{\Omega}\, X$ Gromov-Witten cycles.

One of the motivations we had for undertaking study of configuration space of smooth stable maps, is so we could make the following definition, we say more in the remark below.

Definition 1.4. *We will say that a symplectic manifold (X, ω) is* **q-complete** *if homology of $\overline{\Omega}\, X$ is multiplicatively generated over $\mathbb{Q}$ by homology classes of Gromov-Witten cycles.*

Remark 1.5. *This is the homological version of homotopy approximation of $\Omega^2 X$ by holomorphic mapping spaces from $\mathbb{CP}^1$, which was studied for example in [1], [6], [5], [8], [4]. Given a symplectic manifold X, ω, the basic question is when does*

$$\Omega^2 X \simeq Hol^+(\mathbb{CP}^1, X, \omega, j_X)), \tag{1.3}$$

where Hol^+ denotes the group completion of the topological monoid of based j_X-holomorphic maps of $\mathbb{CP}^1$ into X, under the gluing operation. Remarkably, this is known to be the case for example, for complex projective spaces, generalized flag manifolds and toric manifolds. Unfortunately (1.3) only makes sense for a fixed complex structure j_X i.e. it is a priori not a symplectic property. We wanted a purely symplectic notion, and q-complete symplectic manifolds is one possibility.

Question 1.6. *Is $\mathbb{CP}^n, \omega_{st}$ q-complete?*

This appears to be at the moment a difficult and interesting question. Some interesting and possibly related work is done by Miller [11].

Acknowledgements. I am grateful to Weimin Chen and Dusa McDuff for discussions.

2. Main argument

2.1. **Quantum classes.** We are going to use the basic notation and definitions of [14]. We will be concerned with stable quantum classes, which are classes

$$qc_k^\infty \in H^{2k}(\Omega^2 BSU(r) \simeq \Omega SU(r), QH(\mathbb{CP}^{r-1})), \tag{2.1}$$

with $2k \leq 2r-2$. The quantum homology ring $QH(\mathbb{CP}^{r-1})$ is taken with $\mathbb{Q}$ coefficients, so that additively it is just the rational homology of $\mathbb{CP}^{r-1}$. The above classes are interesting in the present context because they canonically "extend" to $\overline{\Omega}\, BSU(r)$, while such an extension is not apparent for Chern classes especially since they are not *intrinsically* defined on $\Omega^2 BSU(r)$. Extend here means that they are pull-backs of some orbifold cohomology classes on $\overline{\Omega}\, BSU(r)$, via the natural map $\Omega^2 BSU(r) \to \overline{\Omega}\, BSU(r)$. Although orbifold cohomology groups for us will just be the space of $\widehat{\mathrm{QH}}(\mathbb{CP}^\infty)$ valued linear functionals on certain orbifold bordism groups $Cob_*^{orb}(\overline{\Omega}\, BSU(r))$, which contain too much information to be really practical, and in principle it would be good to cut this down, but it is not obvious how to do this in a way that still allows definition of our classes. In particular it is at the moment unclear how and if these are related to Chen-Ruan orbifold cohomology groups (with suitable coefficients). A map $X \to \overline{\Omega}\, BSU(r)$ by definition factors through a map $X \to \overline{\Omega}_n\, BSU(r)$ for some n, composed with the universal map

$$\overline{\Omega}_n\, BSU(r) \to \overline{\Omega}\, BSU = colim_n\, \overline{\Omega}_n\, BSU(r).$$

Consequently, we may define a map from a smooth manifold $f : X \to \overline{\Omega}\, BSU(r)$ as sc-smooth if factors through an sc-smooth map to $\overline{\Omega}_n\, BSU(r)$. Note that $BU(r) = \lim_{n \mapsto \infty} Gr_{\mathbb{C}}(r, \mathbb{C}^n)$, and we of course only need a polyfold structure on $\overline{\Omega}_n\, Gr_{\mathbb{C}}(r, \mathbb{C}^n)$ for every n, for all our arguments. To simplify notation we will still just refer $\overline{\Omega}\, BSU(r)$ as a polyfold, although this of course only means that it is a direct limit of polyfolds.

The group $Cob_k^{orb}(\overline{\Omega}\, BSU(r))$, is the group of equivalence classes of sc-smooth orbifold maps $f : X^k \to \overline{\Omega}\, BSU(r)$, with X^k closed oriented smooth orbifold with corners, with composition given by disjoint union, where the equivalence relation is $(f_1, X_1) \sim (f_2, X_2)$ if there is an sc-smooth map of an orbifold with boundary:

$$F : B^{k+1} \to \overline{\Omega}\, BSU(r),$$

with $(\partial F, \partial B) = (f_1, X_1^{op}) \sqcup (f_2, X_2)$, where X_1^{op} denotes X_1 with the opposite orientation. From the groupoid point of view of orbifolds an orbifold X is some coarse equivalence class of an ètale, proper, stable, smooth groupoid, also known as orbifold groupoid, $\mathcal{X}$, see for example [9]. The orbifold map f above is determined by data of an sc-smooth functor $\widetilde{f} : \mathcal{X} \to \overline{\Omega}\, BSU(r)$, with the right side considered as a polyfold groupoid, also called ep-groupoid, but we only use the former name. Let $E \to BSU(r)$ denote the projectivization of the universal $\mathbb{C}^r$-bundle.

We have a natural polyfold groupoid fibration $\mathcal{E} \to \overline{\Omega}\, BSU(r)$, with fiber over $x \in \overline{\Omega}\, BSU(r)$, $[x : \Sigma_x \to BSU(r)]$ the space of "stable sections" i.e. section class stable maps $\sigma : \Sigma'_x \to E_x = x^*E$. These are defined as follows: Σ'_x has some components labeled as principal, and some components labeled as vertical. The principal components of Σ'_x are identified with components of Σ_x and σ is a "smooth" (continuous section with appropriate regularity [7, Section 1.1]) section of E_x over these components, while the vertical components of Σ'_x are mapped into

the fibers of E_x. The restriction of σ to the collection of all vertical components is required to be a stable map as defined in Section 1.1.

This definition requires a bit more explanation as x is only some equivalence class of a map. To this end consider the large category $\mathcal{P}^A_{0,n}BSU(r)$ with morphisms given by label preserving reparametrizations, (see Section 1.1). Then clearly we have an analogously defined fibration of topological categories $\mathcal{E} \to \mathcal{P}^A_{0,n}BSU(r)$ (and now there is no ambiguity in the definition). On the other hand the groupoid $\overline{\Omega}\, BSU(r)$ is actually constructed as a refinement of $\mathcal{P}^A_{0,n}BSU(r)$ and this refinement also refines the above categorical fibration to a polyfold groupoid fibration $pr : \mathcal{E} \to \overline{\Omega}\, BSU(r)$. In what follows we keep applying this implicit understanding.

Fix a unitary (in other words $PU(r)$) connection $\mathcal{A}$ on $E \to BSU(r)$. Then the restriction $\mathcal{A}_x$ of $\mathcal{A}$ to $E_x = x^*E$ induces almost complex structures $\{J_x\}$ in the following standard way.

- The natural map $\pi : (E_x, J_x) \to (\Sigma_x, j_x)$ is holomorphic.
- J_x preserves the horizontal subbundle of TE_x induced by $\mathcal{A}_x$.
- J_x preserves the vertical tangent bundle $T^{vert}E_x$ of $\mathbb{CP}^{r-1} \hookrightarrow E_x \to \Sigma_x$, and restricts to the standard complex structure on the fibers $\mathbb{CP}^{r-1}$. (That is to say the fibers are identified with $\mathbb{CP}^{r-1}$ up to action of $PU(r)$, which preserves this complex structure.)

Let $\widetilde{\mathcal{E}}$ denote the pull-back of $\mathcal{E}$ to $\mathcal{X}$, via $\widetilde{f} : \mathcal{X} \to \overline{\Omega}\, BSU(r)$. Then over $\widetilde{\mathcal{E}}$ we have a natural strong Polyfold Banach bundle $\mathcal{W}$. For an element $\sigma \in E_x$, $\sigma : \Sigma'_x \to E_x$, the fiber over σ consists (after appropriate completion) of the space of continuous, and smooth over smooth components J_x anti complex linear 1-forms on Σ'_x with values in $\sigma^* T^{vert}E_x$. By essentially identical arguments to [7, Section 1.2] over the whole $\mathcal{E}$ this can be given the structure of a strong polyfold Banach bundle. And we have a Fredholm sc-section of $\mathcal{W}$: the Cauchy-Riemann section, by taking the J_x anti-linear part of the differential of $\sigma = (\Sigma'_x \to E_x)$ as a map into $\sigma^* T^{vert}E_x$. We finally define our orbifold cohomology classes

$$qc_k \in H^k(\overline{\Omega}\, BSU(r), QH(\mathbb{CP}^{r-1})),$$

which to remind the reader for us are just $QH(\mathbb{CP}^{r-1})$ valued linear functionals on $Cob^{orb}_k(\overline{\Omega}\, BSU(r))$. We will restrict our discussion to the identity component of $\overline{\Omega}\, BSU(r)$. We need to say how to compute

$$\langle qc_k, [\widetilde{f}]\rangle, \tag{2.2}$$

where $\widetilde{f}$ denotes the functor $\widetilde{f} : \mathcal{X} \to \overline{\Omega}\, BSU(r)$ lifting the data of an sc-smooth orbifold map $f : X \to \overline{\Omega}\, BSU(r)$.

Recall that each Σ_x comes with a distinguished marked point z_0, which is mapped to a fixed base point in $BSU(r)$, (recall definition of $\overline{\Omega}\, BSU(r)$). Consequently, we have a smooth family of embeddings $I_x : \mathbb{CP}^{r-1} \to E_x$, which takes $\mathbb{CP}^{r-1}$ to the fiber of E_x over z_0. Let $\overline{\mathcal{M}}(\mathcal{P}, [\mathbb{CP}^l], d)$ denote the polyfold groupoid consisting of pairs (σ, x), $x \in \mathcal{X}$ with $\sigma \in \widetilde{\mathcal{E}}$ in the 0-set of the above constructed Cauchy-Riemann section, whose total degree is d, and which intersects $I_x(\mathbb{CP}^l)$. Where d here is defined as

$$\langle c_1(T^{vert}E_x), \sigma\rangle \cdot \frac{1}{r}.$$

This is an integer, since by assumption $\widetilde{f}$ maps to the identity component of $\overline{\Omega}\, BU(r)$ which implies that $P_x \simeq \mathbb{CP}^{r-1} \times \Sigma_x$, as a topological bundle. But then

the vertical Chern number of a section is n times the degree of the projection of the section to $\mathbb{CP}^{r-1}$.

The virtual dimension of $\overline{\mathcal{M}}(\mathcal{P}, [\mathbb{CP}^l], d)$, is $2dr + 2k + 2l$. Note that if $0 < 2k \leq 2r - 2$, then

$$2dr + 2k + 2l < 0 \text{ unless } d \geq -1.$$

On the other hand $d > 0$ results in too high virtual dimension, and $d = 0$ only contributes to degree 0 class as we will see below. Under this condition definition of quantum classes is particularly simple:

$$\langle qc_k, [\widetilde{f}]\rangle = b \in QH(\mathbb{CP}^{r-1}),$$

where $b \in H_*(\mathbb{CP}^{r-1}, \mathbb{Q})$ is defined by duality:

$$b \cdot [\mathbb{CP}^l] = \#\overline{\mathcal{M}}(\mathcal{P}, [\mathbb{CP}^l], -1) \in \mathbb{Q},$$

where the left side is the usual intersection product and the right side is the orbifold Gromov-Witten invariant counting elements of $\overline{\mathcal{M}}(\mathcal{P}, [\mathbb{CP}^l], -r)$, which is zero unless the expected dimension is zero. Of course we also have to show that our definition of qc_k is well defined, i.e. that (2.2) is independent of the choice of representative $\widetilde{f}$. This is worked out in [13], and readily generalizes to polyfold setting, as it is just the usual cobordism of moduli space argument, similarly with independence of choices of almost complex structures, (connections).

The pull-back of the above defined classes to $\Omega^2 BSU(n)$ are exactly the stable quantum classes considered in [14], except of course we did not need orbifold cycles but worked with cycles that are maps of closed oriented manifolds. We paraphraze the main theorem of [14] as follows:

Theorem 2.1 ([14]). *If $2k \leq 2r - 2$, then $0 \neq a \in H_{2k}(\Omega^2 BSU(r), \mathbb{Q})$ if and only if for some $\{\beta_i, \alpha_i\}$*

$$0 \neq \langle \prod_i qc_{\beta_i}^{\alpha_i}, a\rangle, \text{ where} \sum_i 2\beta_i \cdot \alpha_i = k.$$

Proof of Theorem 1.3. By Milnor-Moore, Cartan-Serre theorems the rational homology of $\Omega^2 BSU(r) \simeq \Omega SU(r)$ is multiplicatively generated by spherical classes, in particular by maps of smooth manifolds. Consequently the theorem follows immediately by Theorem 2.1 and by existence of extension of classes qc_k to $\overline{\Omega}\, BSU(r)$ described above. ☐

References

[1] R. L. Cohen, J. D. S. Jones, and G. B. Segal, *Stability for holomorphic spheres and Morse theory*, Contemporary Mathematics, (2000), pp. 87 – 106.

[2] J. Eells and J. Wood, *Maps of minimum energy.*, J. Lond. Math. Soc., II. Ser., 23 (1981), pp. 303–310.

[3] H. Federer, *Geometric measure theory*, no. 153 in Die Grund. der math. Wissenschaften, Springer, New York, 1969.

[4] J. Gravesen, *On the topology of spaces of holomorphic maps.*, Acta Math., 162 (1989), pp. 247–286.

[5] M. Guest, *Topology of the space of absolute minima of the energy functional.*, Am. J. Math., 106 (1984), pp. 21–42.

[6] M. A. Guest, *The topology of the space of rational curves on a toric variety.*, Acta Math., (1995).

[7] H. Hofer, W. Kris, and E. Zehnder, *Applications of Polyfold Theory I: The Polyfolds of Gromov-Witten Theory*, arXiv:1107.2097, (2011).

[8] F. Kirwan, *On spaces of maps from Riemann surfaces to Grassmannians and applications to the cohomology of moduli of vector bundles.*, Ark. Mat., 24 (1986), pp. 221–275.

[9] D. McDuff, *Groupoids, branched manifolds and multisections.*, J. Symplectic Geom., 4 (2006), pp. 259–315.

[10] D. McDuff and D. Salamon, *J–holomorphic curves and symplectic topology*, no. 52 in American Math. Society Colloquium Publ., Amer. Math. Soc., 2004.

[11] J. Miller, *Homological stability properties of spaces of rational J-holomorphic curves in* P^2, arXiv:1110.1665, (2011).

[12] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres.*, (1981).

[13] Y. Savelyev, *Quantum characteristic classes and the Hofer metric*, Geometry Topology, (2008).

[14] ———, *Bott periodicity and stable quantum classes*, arXiv:0912.2948, (2012).

Centre de Recherches Mathématiques, Université de Montréal, C.P. 6128, Succ. Centre-ville, Montréal H3C 3J7, Québec, Canada

E-mail address: savelyev@crm.umontreal.ca